(Problem written by team 5. Based on problem 8.6 in Schaum's QM)

Consider a particle in a central field and assume that the system has a discrete spectrum. Each orbital quantum number l has a minimum energy value. Show that this minimum value increases as l increases.

We begin by writing the Hamiltonian of the system.

$H = \frac{-\hbar^2}{2mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} + V(r)$

Using $H_1 = \frac{-\hbar^2}{2mr^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + V(r)$ we have that

$H = H_1 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$

The minimum value of the energy in the state l is

$E^l_{min} = \int \psi^*_l [H_1 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]\psi_l \, \mathrm{d}^3r$

The minimum value of the energy in the state $l+1$ is given by

$E^{l+1}_{min} = \int \psi^*_{l+1} [H_1 + \frac{\hbar^2}{2m}\frac{(l+1)(l+2)}{r^2}]\psi_{l+1} \, \mathrm{d}^3r$

This equation for the $l+1$ state can then be written in the form

$E^{l+1}_{min} = \int \psi^*_{l+1} \frac{\hbar^2}{m}\frac{l+1}{r^2}\psi_{l+1} \, \mathrm{d}^3r + \int \psi^*_{l+1} [H_1 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]\psi_{l+1} \, \mathrm{d}^3r$

Since $\mid \psi_{l+1} \mid^2$ and $\frac{\hbar^2}{m}\frac{l+1}{r^2}$ are positive, the second term in this equation is always positive. Consider now the first term. ψ_l is an eigenfunction of the Hamiltonian $H = H_1 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$ and corresponds to the minimum eignevalue of this hamiltonian. Thus,

$\int \psi^*_l [H_0 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]\psi_l \, \mathrm{d}^3r < \int \psi^*_{l+1} [H_0 + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}]\psi_{l+1} \, \mathrm{d}^3r$

This proves that $E^l_{min} < E^{l+1}_{min}$