# Properties of Astrophysical Plasmas

- Basic Thermodynamics for Quantum Systems

- Distribution functions

- Thermodynamical Variables and Potentials

- Equations of State (General & limiting cases)

## Why all the efforts ?

Results of nuclear processes depend on
- how the object (star, universe) reacts to nuclear reactions and
- on the competition between microscopic, nuclear reactions, and macroscopic, hydro and transport, system.

# Basic Thermodynamical Quantities

Particle density
$$n = \frac{N}{V} = \int_0^\infty \omega(p) f(p) dp$$

Energy density
$$u = \frac{U}{V} = \int_0^\infty E\omega(p) f(p) dp$$

Pressure
$$P = \frac{1}{3}\int_0^\infty pv\omega(p) f(p) dp$$

with
v: velocity, p: momentum, E: energy of state,

occupation probability $f(p)$,

state density $\omega(p)$ per unit volume

# Quantum Dynamical Thermodynamical System: Stationary Schroedinger Equation

$$H\psi_{1,\ldots n} = (\mathcal{T} + \mathcal{V})\psi_{1,\ldots n} = E\psi_{1,\ldots n}$$

kinetic E.

$$\mathcal{T} = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} -\frac{\hbar^2}{2m}\Delta_i$$

$$\mathcal{V} = \sum_{i=1}^{n} \sum_{j>i}^{n} \mathrm{v}_{ij}.$$

potential

$\left[-\frac{\hbar^2}{2m}\Delta + \mathrm{v}\right]\phi = \varepsilon\phi$ spatial probability function of all particles

**For non-interacting Particles:**

**t(i,j)=0 and v(i,j) = 0**

**Equations are separable !!!**

# State Density for non-interacting Gas

$$\left[-\frac{\hbar^2}{2m}\Delta + \mathrm{v}\right]\phi = \varepsilon\phi \qquad \phi = X \cdot Y \cdot Z,$$

thus (see deviation on blackboard)

Eigenvalues in energy in x,y,z $$\varepsilon_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2md^2}(n_x^2 + n_y^2 + n_z^2) = \frac{\pi^2\hbar^2}{2md^2}R_\rho^2.$$

Integrate over all points within radius R for an octant (factor 1/8):

Number of states $$\Phi(E) = \frac{4\pi}{3}\frac{g}{h^3}(2m)^{3/2}E^{3/2}$$

State density: $$\omega(E) = 2\pi\frac{g}{h^3}(2m)^{3/2}E^{1/2}.$$

**Remark: E = $p^2$/2m** Fermi-Energy: All level filled

# Occupation Probabilities

$$f(p) = \begin{cases} [e^{(E(p)-\mu)/kT} + 1]^{-1} & \text{Fermions} \\ [e^{(E(p)-\mu)/kT} - 1]^{-1} & \text{Bosons} \\ e^{-(E(p)-\mu)/kT} & \text{Maxwell-Boltzmann} \end{cases}$$

$\mu$ stands for the chemical potential without restmass. $\bar{\mu}$,

For equilibrium reactions between 2 states with C1/2/3/4 particles, we have

$$C_1\bar{\mu}_1 + C_2\bar{\mu}_2 = C_3\bar{\mu}_3 + C_4\bar{\mu}_4$$

$$\bar{\mu} = \mu + mc^2.$$

# Thermodynamical Variables and Potentials

| thermodynamic variable | differential |
|---|---|
| heat $Q(= q \cdot V)$ | $dQ = TdS$ |
| energy $U(= u \cdot V)$ | $dU = TdS - PdV + \sum_i \bar{\mu}_i dN_i$ |
| entropy $S(= s \cdot V)$ | $dS = \frac{1}{T}dU + \frac{P}{T}dV - \sum_i \frac{\bar{\mu}_i}{T} dNi$ |
| free energy $F = U - TS$ | $dF = -SdT - PdV + \sum_i \bar{\mu}_i dN_i$ |
| enthalpy $H = U + PV$ | $dH = TdS + VdP + \sum_i \bar{\mu}_i dN_i$ |
| free enthalpy $G = U - TS + PV$ | $dG = -SdT + VdP + \sum \bar{\mu}_i dN_i$ |
| thermodynamic potential $\Omega = U - TS - \sum_i \bar{\mu}_i N_i$ | $d\Omega = -SdT - PdV - \sum_i N_i d\bar{\mu}_i$ |

# The Equation of State (non-interacting)

Remark: This Formalism allows us to calculate general EOS, of non, partial, or fully relativistic gases, and to recover limiting

$$n = \sum_i n_i$$

$$P = \sum_i P_i$$

$$n_i = \frac{4\pi g_i}{h^3} \int_0^\infty \frac{p_i^2 dp_i}{\exp[\frac{m_i c^2}{kT}\left(1 + \frac{p_i^2}{m_i^2 c^2}\right)^{1/2} - \frac{\bar{\mu}_i}{kT}] \pm 1}$$

$$P_i = \frac{4\pi c^2 g_i}{3h^3} \int_0^\infty \frac{p_i^4 dp_i}{m_i c^2 \left(1 + \frac{p_i^2}{m_i^2 c^2}\right)^{1/2} [\exp(\frac{m_i c^2}{kT}\left(1 + \frac{p_i^2}{m_i^2 c^2}\right)^{1/2} - \frac{\bar{\mu}_i}{kT}) \pm 1]},$$

or with $E_i = \sqrt{p_i^2 c^2 + (m_i c^2)^2}$

$$n_i = \frac{4\pi g_i}{h^3} \int_0^\infty \frac{p_i^2 dp_i}{\exp[\frac{E_i}{kT} - \frac{\bar{\mu}_i}{kT}] \pm 1}$$

$$P_i = \frac{4\pi c^2 g_i}{3h^3} \int_0^\infty \frac{p_i^4 dp_i}{E_i[\exp(\frac{E_i}{kT} - \frac{\bar{\mu}_i}{kT}) \pm 1]}.$$

# Examples: Photon Gas (boson, spin=1)

**Massless Boson**
(Only possible without chemical potential)

$$dQ = TdS = 0 = dU + pdV - \bar{\mu}dN.$$

**THUS: P=1/3 u**

$$\begin{aligned} n &= \int_0^\infty 4\pi \frac{2}{h^3} \left(\frac{h\nu}{c}\right)^2 \frac{1}{e^{h\nu/kT} - 1} \frac{h}{c} d\nu \\ &= \int_0^\infty \frac{8\pi\nu^2}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \end{aligned}$$

$$n = \frac{2.404 \cdot 8\pi k^3}{h^3 c^3} T^3$$

$$\begin{aligned} u &= \int_0^\infty h\nu \ 4\pi \frac{2}{h^3} \left(\frac{h\nu}{c}\right)^2 \frac{1}{e^{h\nu/kT} - 1} \frac{h}{c} d\nu \\ &= \int_0^\infty \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT} - 1} d\nu \end{aligned}$$

$$u = \frac{8\pi^5 k^4}{15 h^3 c^3} T^4 = aT^4$$

$$P = \frac{1}{3} \int_0^\infty 4\pi \frac{2}{h^3} \left(\frac{h\nu}{c}\right)^2 \frac{1}{e^{h\nu/kT} - 1} c \frac{h\nu}{c} \frac{h}{c} d\nu$$

# Examples: Boltzmann Gas

$$n = \int_0^\infty 4\pi \frac{g}{h^3} p^2 e^{-(E-\mu)/kT}\, dp$$

$$E = \frac{1}{2} m v^2 = \frac{p^2}{2m}$$

$$n = e^{\mu/kT} 4\pi \frac{g}{h^3} \int_0^\infty p^2 e^{-\frac{p^2}{2mkT}}\, dp$$

$$= e^{\mu/kT} \frac{g}{h^3} (2\pi mkT)^{3/2}$$

$$\Rightarrow \bar{\mu} = \mu + mc^2 = kT \ln\left(\frac{nh^3}{g} \frac{1}{(2\pi mkT)^{3/2}}\right) + mc^2$$

$$u = \int_0^\infty E \frac{n 4\pi p^2}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}}\, dp$$

$$\left[\int_0^\infty p^4 e^{-\frac{p^2}{2mkT}}\, dp = \frac{3}{8}\sqrt{\pi}(2mkT)^{5/2}\right]$$

$$u = \frac{3}{2} nkT$$

$$P = \frac{1}{3} \int_0^\infty pv \frac{n 4\pi p^2}{(2\pi mkT)^{3/2}} e^{-\frac{p^2}{2mkT}}\, dp$$

$$P = nkT$$

# Examples: Non-rel., degenerate Fermi Gas I

$$n = \frac{4\pi(2m)^{3/2}}{h^3} \int_0^\infty \frac{E^{1/2}}{\exp[(E-\mu)/kT]+1} dE.$$

We introduce a coordinate transformation $x = E/kT$, $dE = kTdx$, $E^{1/2} = (kT)^{1/2}x^{1/2}$ and a Fermi integral

$$n = 4\pi \frac{(2mkT)^{3/2}}{h^3} \int_0^\infty \frac{x^{1/2}}{\exp(x - \frac{\mu}{kT})+1} dx.$$

In general the Fermi functions $F_n(z)$ are defined by

$$F_n(z) = \int_0^\infty \frac{x^n dx}{1+e^{x+z}}$$

$$\Rightarrow n = 4\pi \frac{(2mkT)^{3/2}}{h^3} F_{1/2}(-\frac{\mu}{kT}).$$

# Examples: Degenerate, non-rel. Fermi Gas II

$$P = \frac{1}{3}\frac{4\pi(2m)^{3/2}}{h^3}\int_0^\infty \frac{2E^{3/2}}{\exp((E-\mu)/kT)+1}dE$$

$$= \frac{8\pi}{3}\frac{(2m)^{3/2}kT^{5/2}}{h^3}\int_0^\infty \frac{x^{3/2}}{\exp(x-\frac{\mu}{kT})+1}dx$$

$$P = \frac{8\pi}{3h^3}(2m)^{3/2}(kT)^{5/2}F_{3/2}(-\frac{\mu}{kT})$$

$$P = \frac{2}{3}nkT\frac{F_{3/2}(-\mu/kT)}{F_{1/2}(-\mu/kT)}.$$

# Examples: Degenerate Fermi Gas III

**a)** $\frac{\mu}{kT} \to \infty$: this corresponds to the extremely degenerate case.

$$P = \frac{h^2}{20m}(\frac{3}{\pi})^{2/3}(Y_e N_A)^{5/3}\rho^{5/3}.$$

$$F_{3/2}(-\frac{\mu}{kT}) = \frac{2}{5}(\frac{\mu}{kT})^{5/2}$$

$$F_{1/2}(-\frac{\mu}{kT}) = \frac{2}{3}(\frac{\mu}{kT})^{3/2}.$$

$$F_{3/2}/F_{1/2} = \frac{3}{5}(\frac{\mu}{kT}).$$

**b)** extremely relativistic: $x = p_0/mc \to \infty$

$$P = \frac{2\pi c}{3h^3}p_0^4 = \frac{2\pi c}{3h^3}\left(\frac{3h^3 n}{8\pi}\right)^{4/3}$$

# Summary: Distribution Functions

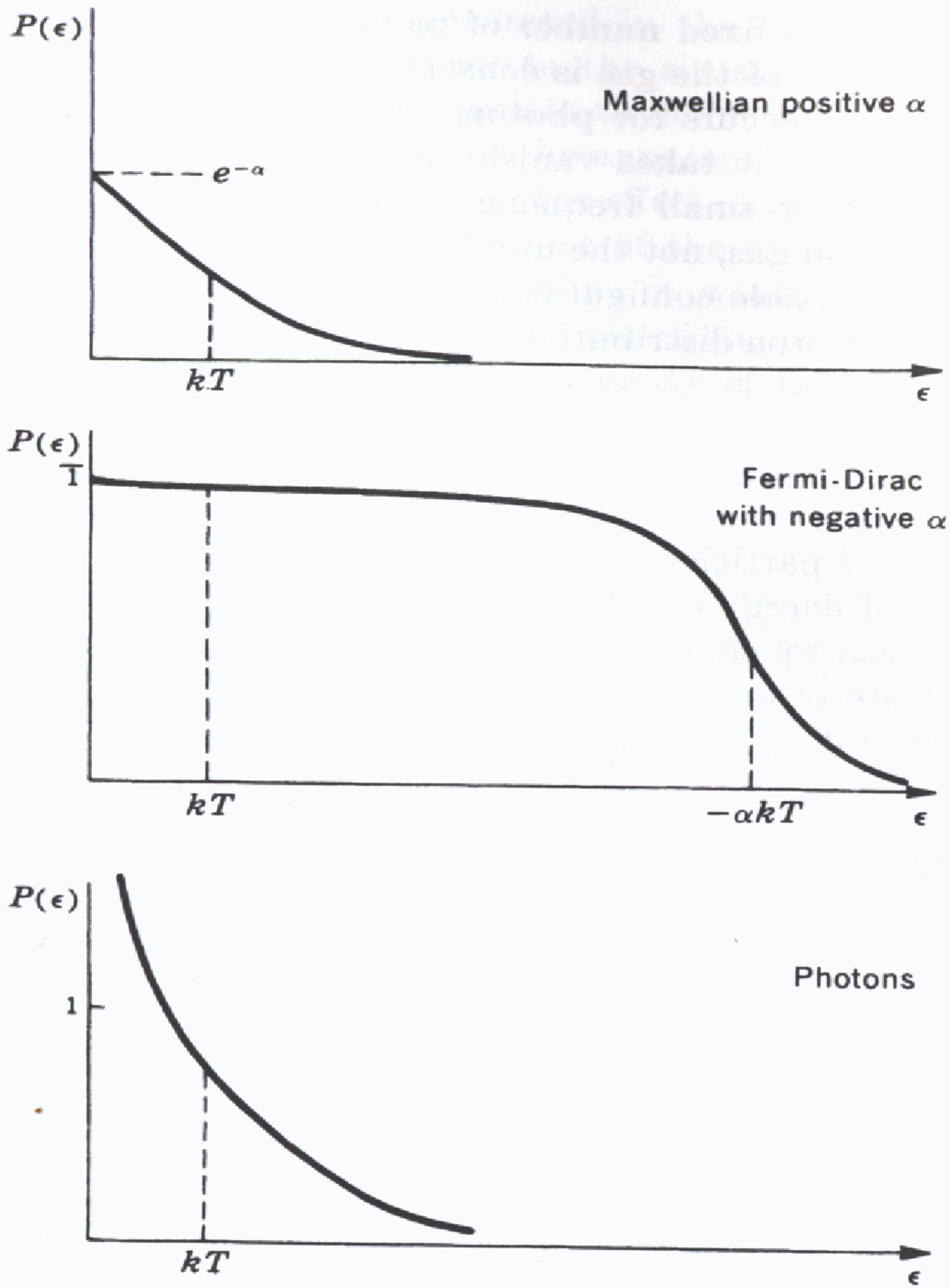
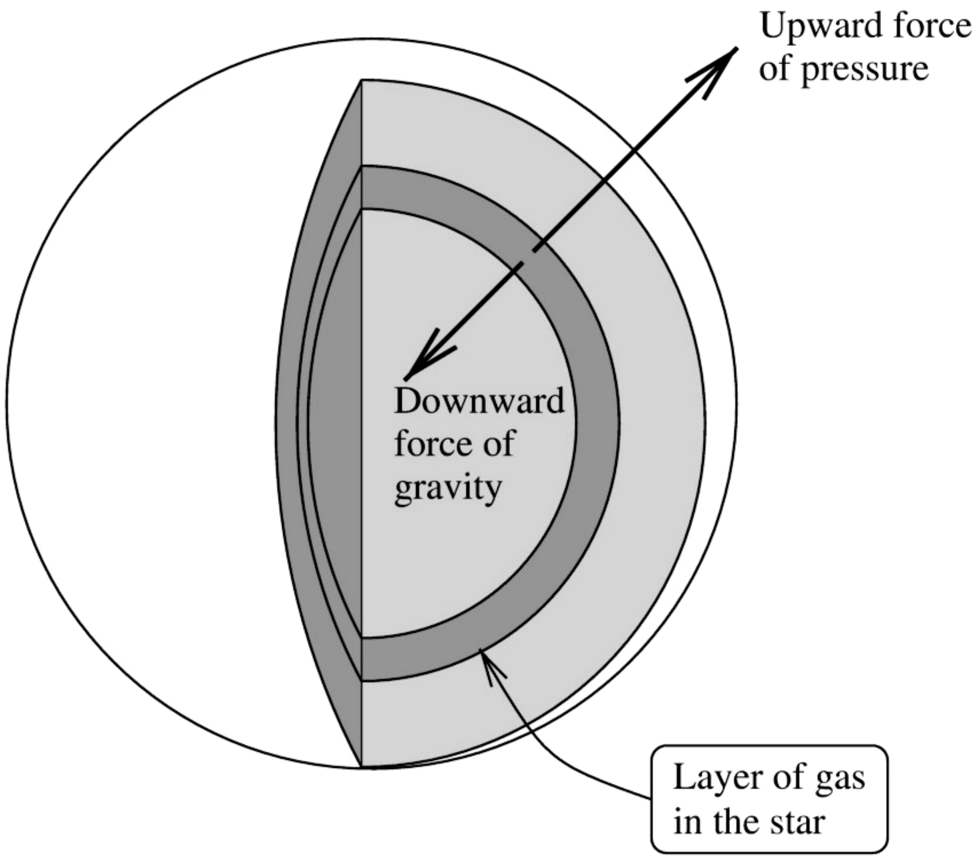



**Fig. 1-3** Schematic occupation index. The Maxwell-Boltzmann distribution, which is a pure exponential curve at all temperatures and all values of $\epsilon$, is valid only if $\alpha \gg 0$. The Fermi-Dirac distribution has, because of the Pauli exclusion principle, an upper bound of unity on the occupation index. This upper bound is approached at low energy and low temperature (or high density). The Einstein-Bose distribution for photons has a large occupation index for $\epsilon \ll kT$ and an exponentially decreasing index for $\epsilon \gg kT$

# Simple Application: Existence of WDs, Neutron Stars, and Black Holes

Cut away view of the interior of a star:




Pressure and gravity must balance or the star will expand or contract.

## Simple Estimates for Stellar Structures

(a) Pressure equilibrium

$$\frac{dP_r}{dr} = -\rho_r \frac{GM_r}{r^2}$$

(b) Conservation of mass

$$\frac{dM_r}{dr} = 4\pi\ r^2 \rho_r$$

Example: dP=P(central)-P(surface), and mean M(r)=M/2 and r=R/2 plus ideal gas for **sun**

P(central) = 7E15 dyn/cm

T(central) < 3E7 K (real value 1.6E7 K)

# Simple Equations of Stellar Structure:
# Polytropic Stars (P=P(rho))

Analytic solutions (of Lane-Emden Equation) for

$$P = \rho^\gamma$$

Case 1) Local by EOS (microscopic equation) / WD and Neutron Star

**Example:** Why do we have an upper limit in Mass for WDs (Chandrasekhar Limit)?

$\bar{\rho} \propto \frac{M_{WD}}{4\pi R^3}$

Gravitational force on mass element: $F_{grav} = \rho \frac{G\ M}{R^2} \propto \frac{M^2}{R^5}$

Gas pressure (EOS) for non-rel. degenerate gas : $P \propto rho^{5/3} \;\Rightarrow dP/dr \propto \frac{M^{5/3}}{R^6}$

Gas pressure (EOS) for rel. degenerate gas : $P \propto rho^{5/3} \;\Rightarrow dP/dr \propto \frac{M^{4/3}}{R^5}$

Non-rel. Gas: Different dependency of EOS and $F_{grav}$ on $\rho \Rightarrow$ star can adjust (stable)

Rel. Gas: Same dependency of EOS and $F_{grav}$ on $\rho \Rightarrow$ star cannot adjust and collapses (unstable)

# Pressure Corrections due to Interactions

(add the term of inner energy)

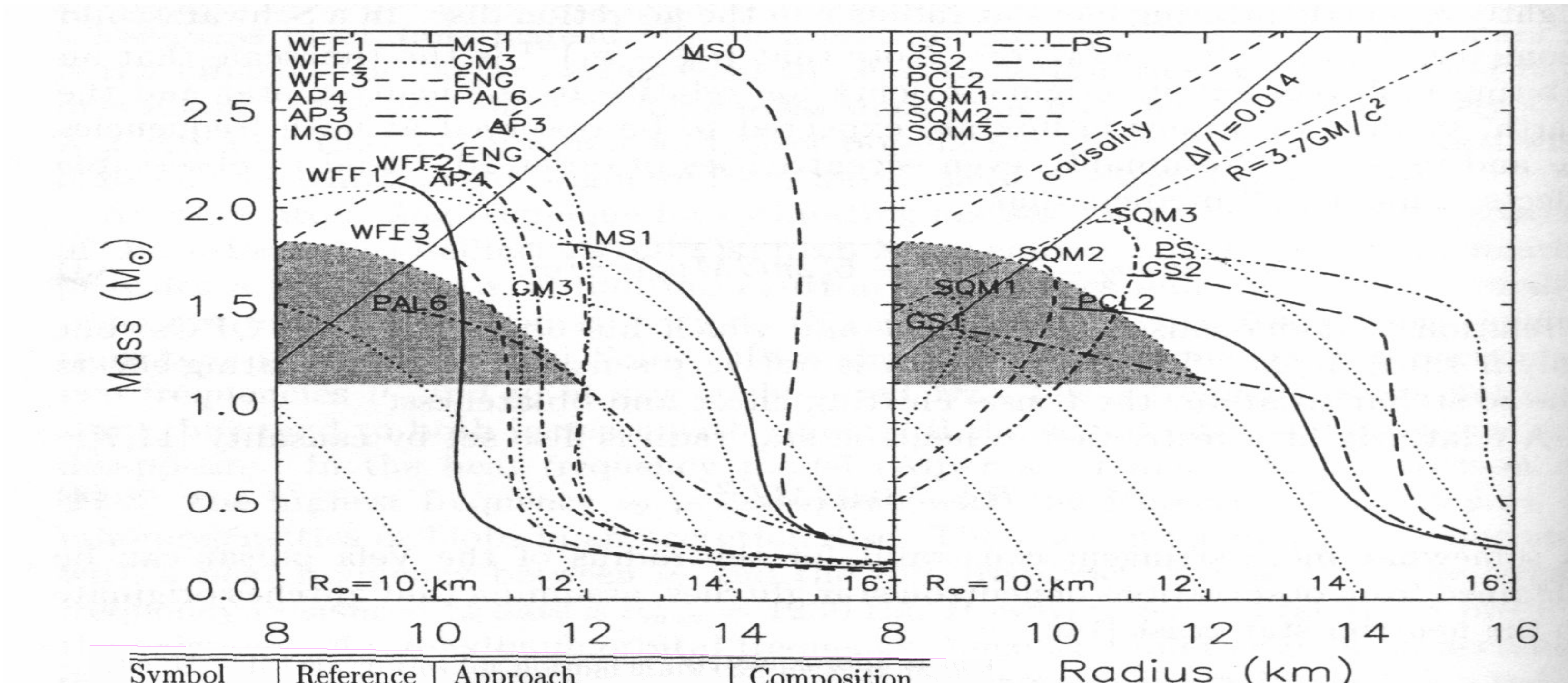
$$U_{tot} = \sum_i U_i + \sum_j U_{int,j}.$$

$$P = -\left(\frac{\partial U_{tot}}{\partial V}\right)_{S,N_i} = \sum_i P_i + \sum_j \left(\frac{\partial U_{int,j}}{\partial V}\right)_{S,N_i}.$$

# Area of Research: Neutron Star vs. Black Holes

The equation of state at high densities (from Lattimer, 2001, AIP 556, 205)

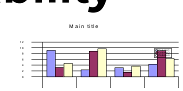
Influence of the EOS on the mass and radius of a neutron star




| Symbol | Reference | Approach | Composition |
|---|---|---|---|
| FP | [5] | Variational | np |
| PS | [20] | Potential | $n\pi^0$ |
| WFF(1-3) | [32] | Variational | np |
| AP(1-4) | [1] | Variational | np |
| MS(1-3) | [15] | Field Theoretical | np |
| MPA(1-2) | [16] | Dirac-Brueckner HF | np |
| ENG | [4] | Dirac-Brueckner HF | np |
| PAL(1-6) | [21] | Schematic Potential | np |
| GM(1-3) | [6] | Field Theoretical | npH |
| GS(1-2) | [8] | Field Theoretical | npK |
| PCL(1-2) | [22] | Field Theoretical | npHQ |
| SQM(1-3) | [22] | Quark Matter | Q $(u, d, s)$ |

- **R(NS) is 11 to 16 km**
- **structure depends on the EOS**
- **M(NS) <1.4 to 2.2 Mo**

**Remark: Tests of QCD and properties far from limit of stability**



# Neutron star masses seem to cluster around 1.4Mo Black hole masses from SRT

| Object[1] | Mass Range (Mo) | Source |
|---|---|---|
| A062000 | 3.3 to 13.6 | Froning et al. 2001 |
| J1118+480 | 6.0 to 7.7? | Wagner et al. 2001 |
| GS J112468 | 4.5 to 6.1 | Bailyn et al. 1998 |
| 4U 154347 | 2.7 to 7.5 | Orosz et al. 1998 |
| GRO J165540 | 5.5 to 7.9 | Shahbaz et al. 1999 |
| J1819.32525 | 8.7 to 11.7 | Orosz et al. 2001 |
| GS 2000+25 | 5.5 to 8.6 | Ioannou et al. 2002 |
| GS 2023+338 | 7.0 to 9.5 | Sanwal et al. 1996 Casares & Charles 1994 |

[1]The masses of J0422+32 and H1705?25 are not reliable but are sometimes included in tables (Rob)

**Remark: Is there a mass gap between NS and BH ?**

**Attention: Possible selection effect for SRTs! The mass gap may reflect the mass of the progenitor at time of the explosion** (if system looses more than ½ of its mass it becomes unbound)