Nuclear Reactions and Cross Sections

- Decays and Transmission Coefficients

- Nuclear Cross Sections

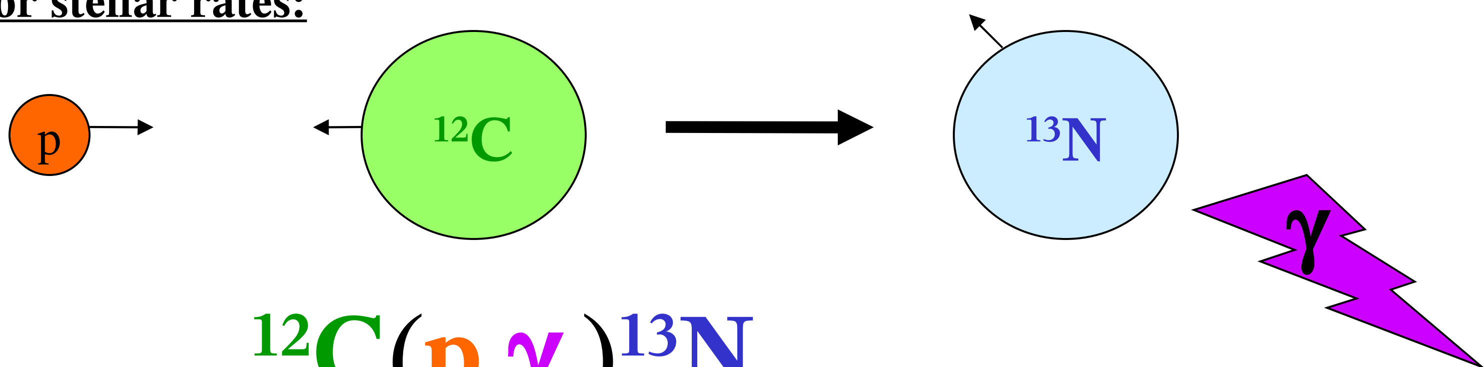
Source: Chapter 2 , and Cameron (1984)

Basic Nomenclature

Nuclear reactions

- generate energy
- create new isotopes and elements

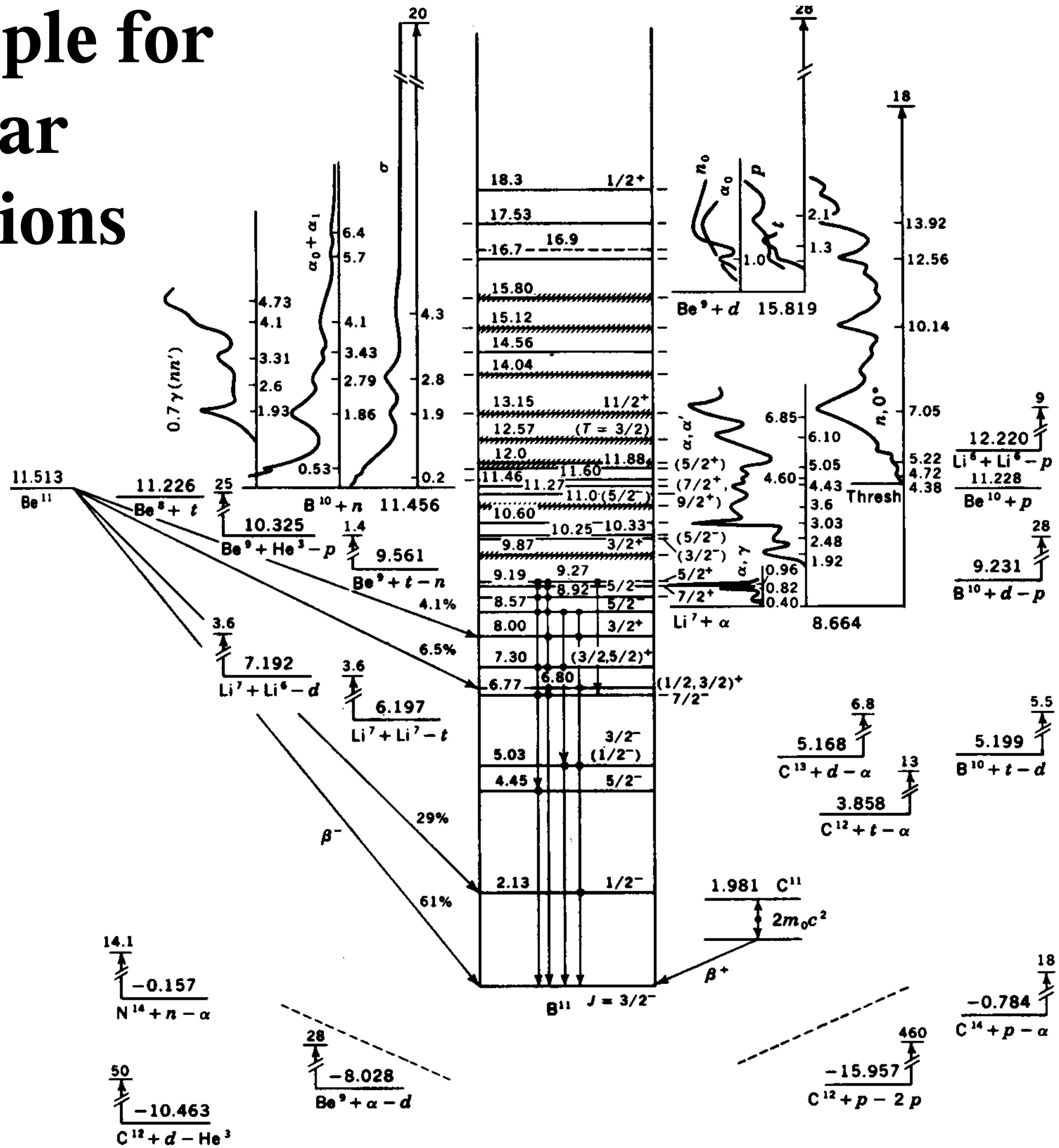
Notation for stellar rates:



$$^{12}C(p,\gamma)^{13}N$$

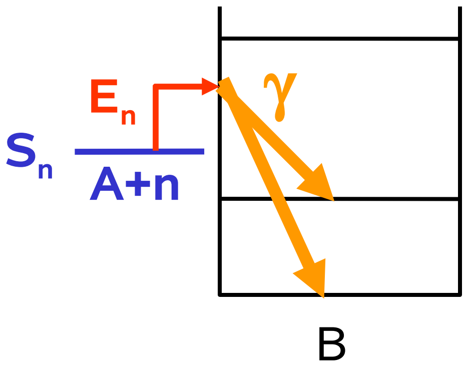
- ^{12}C: The heavier "target" nucleus
- p: the lighter "incoming projectile"
- γ: the lighter "outgoing particle"
- ^{13}N: the heavier "residual" nucleus

Example for Nuclear Reactions



Simplified Example: neutron capture A + n -> B + γ

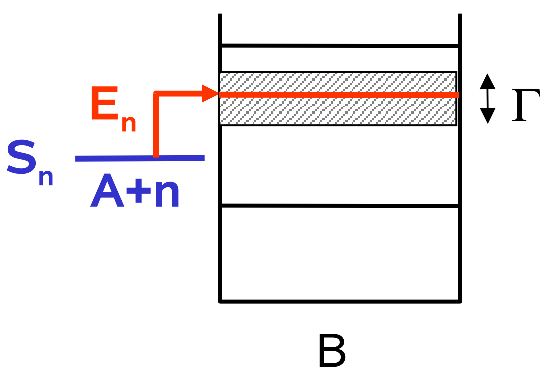
I. **Direct reactions** (for example, direct capture)



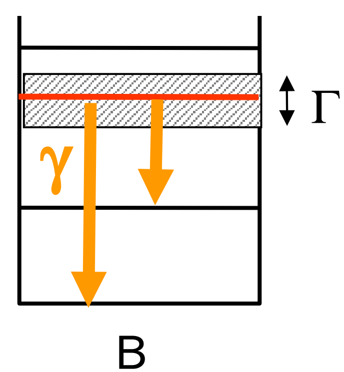
direct transition into bound states

II. **Resonant reactions** (for example, resonant capture)

Step 1: Coumpound nucleus formation
(in an unbound state)

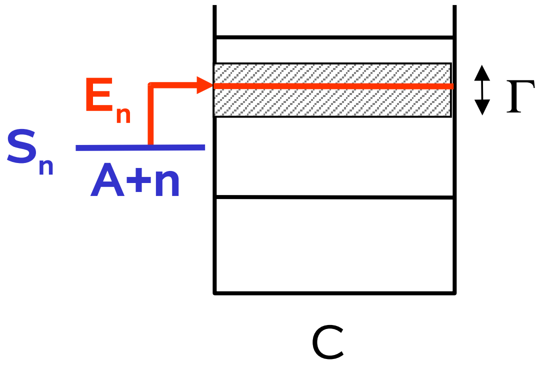


Step 2: Coumpound nucleus decay

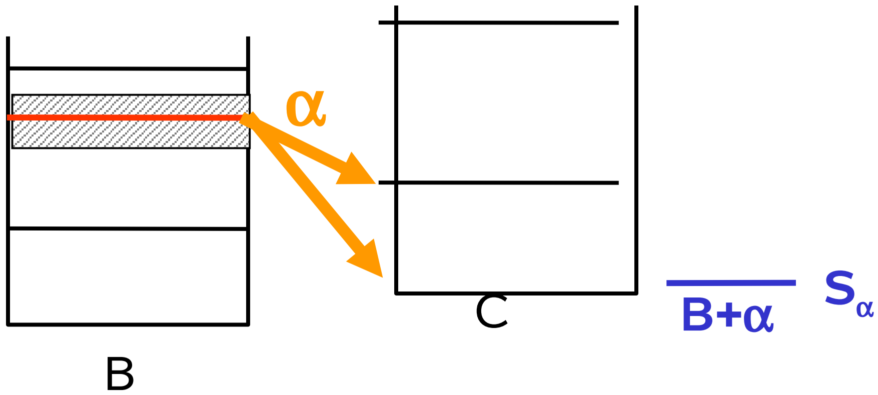


or a resonant A(n,α)B reaction:

Step 1: Compound nucleus formation
(in an unbound state)



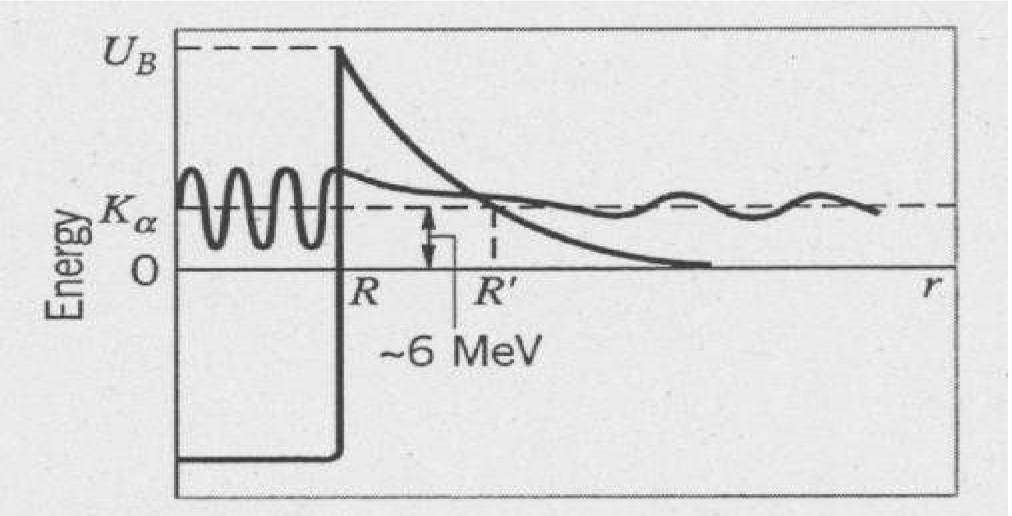
Step 2: Compound nucleus decay



For resonant reactions, E_n has to "match" an excited state (but all excited states have a width and there is always some cross section through tails)

Transmission Coefficients for Coulomb Barriers (or Probabilities for Decays)

Problem:



Ansatz:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right]\phi = E\phi.$$

Solution:

$$\phi = e^{\pm ikx}$$

$$\phi = e^{\pm Kx}$$

$$k = \frac{\sqrt{2m(E-V)}}{\hbar} \text{ for } E > V \qquad K = \frac{\sqrt{2m(V-E)}}{\hbar} \text{ for } E < V.$$

Flux and Transmission Probability

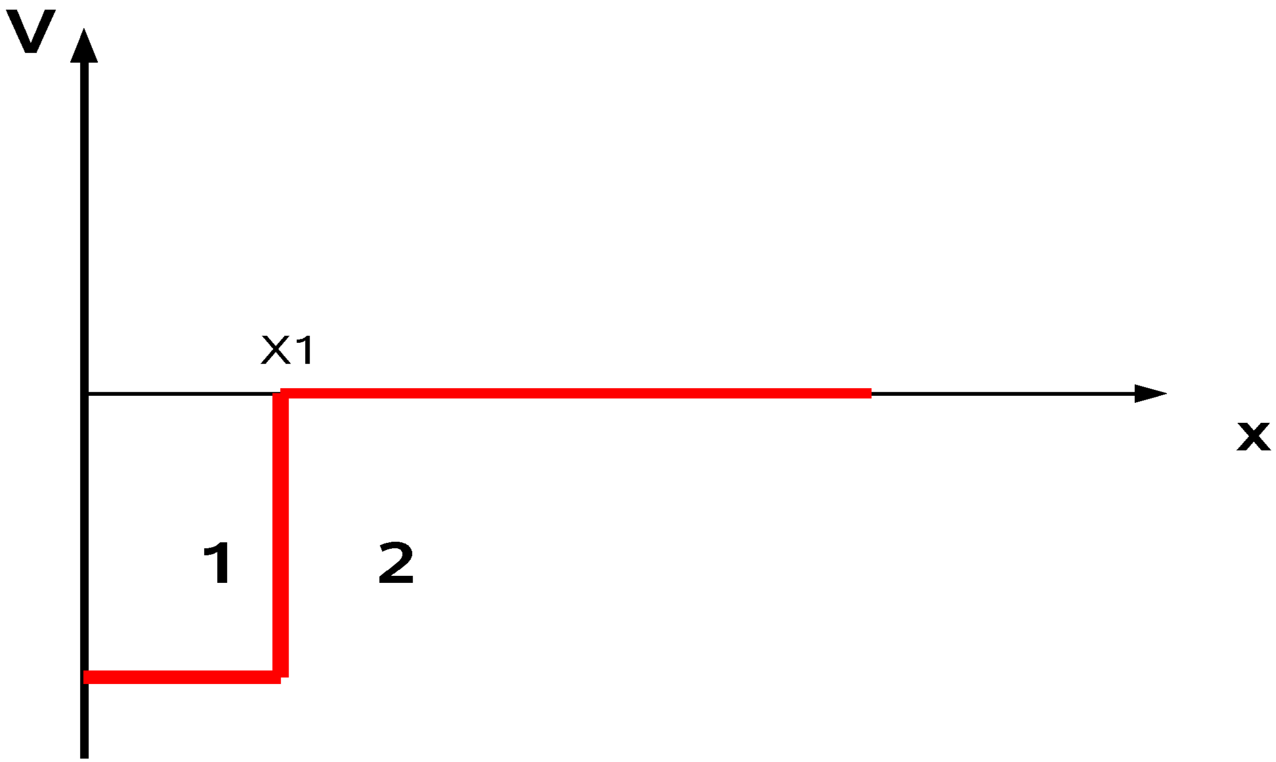
a) Quantum-mechanical Flux

$$\vec{j} = \frac{\hbar}{2mi}(\phi^* \vec{\nabla}\phi - \phi\vec{\nabla}\phi^*).$$

b) Transition Probability

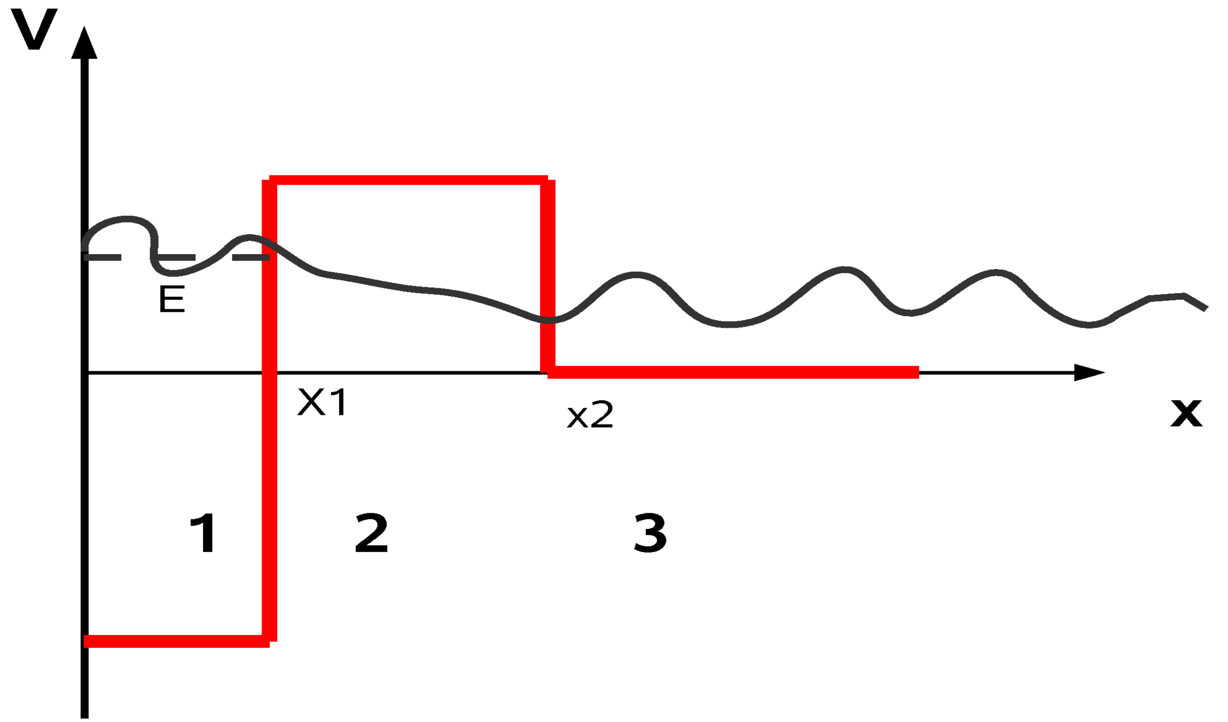
$$T = \frac{j_{fin}}{j_{in}} = \frac{k_{fin}|\phi_{fin}|^2}{k_{in}|\phi_{in}|^2}$$

Example I: Box Potential (n emission)



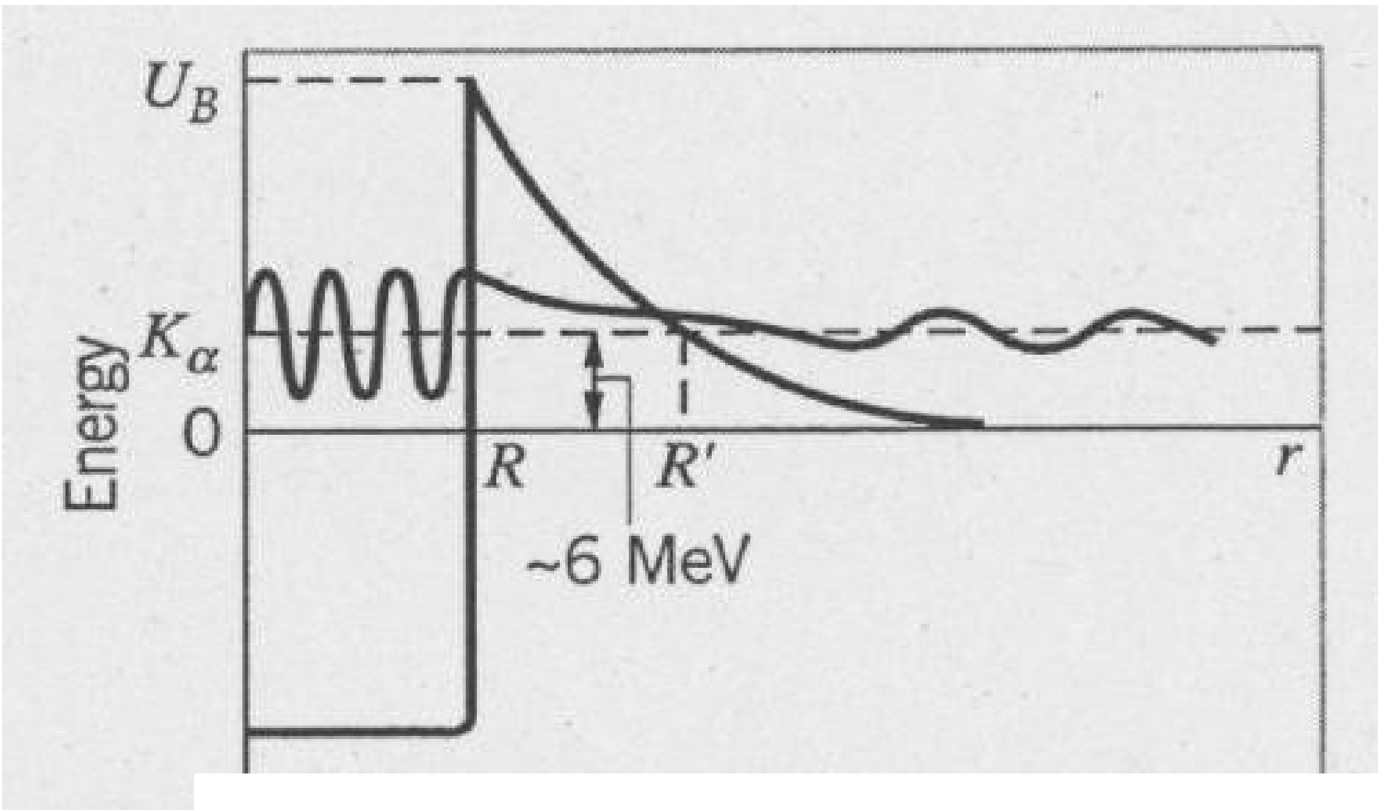
$$T \approx \frac{4k_1}{k_2}.$$

Example II: Box Potential plus square potential (charged particle such as proton)



$$T \propto e^{-\frac{2}{\hbar}\sqrt{2m(V-E)}d}$$

Example IIb: Box plus general Potential V(x)
WKB approximation (Wentzel, Kramers, Brouillon)



$$T \approx e^{-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx}$$

Application to General Decays

Z1: charge of nucleus, Z2: charge of particle with energy E

Coulomb Potential: $V(x)=Z1\ Z2\ e^2 / x$

=>

$$T = e^{-2\pi\eta}$$

$$\eta = \sqrt{\frac{m}{2E}}\frac{Z_1 Z_2 e^2}{\hbar}$$

(Sommerfeld parameter)

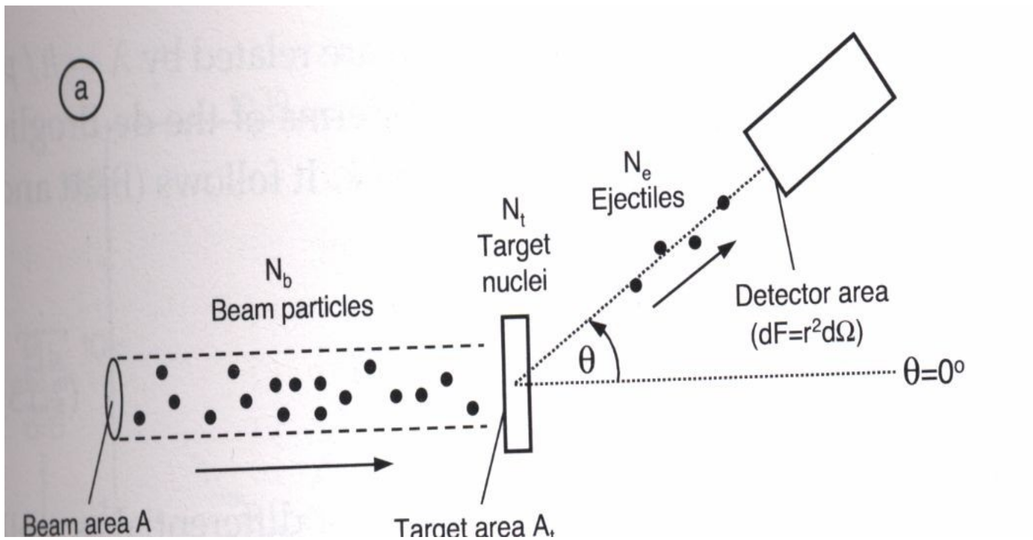
Implications for Astrophysics:

- E > V (explosive burning, nuclear statistical eq.)

- low Z2 processes dominate

Nuclear Cross Sections

Definition: $\sigma = \dfrac{\text{number of reactions per target/sec}}{\text{flux of incoming projectiles}}$



Relation to Tunneling Probablity:

$$\sigma = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) T_l$$

What do we need for the Cross Section?

1) Incoming flux described by plane wave

2) Scattering in all direction

3) Penetration probability into the nucleus.

1) Incoming plane wave (in z-direction)

$$\psi_{in} = e^{ikz} = e^{ikr\cos\theta}$$

$$= \sqrt{4\pi}\sum_{l=0}^{\infty}\sqrt{2l+1}\,i^l j_l(kr)Y_{l,0}(\theta)$$

Remark: Bessel function for large kr

$$j_l(kr) \rightarrow \frac{\sin\left(kr - \frac{l\pi}{2}\right)}{kr} \quad \text{for } r \rightarrow \infty$$

$$\sin x = \frac{i}{2}(e^{-ix} - e^{ix}).$$

$$\psi_{in} = e^{ikz} = \sqrt{\pi}\sum_{l=0}^{\infty}\sqrt{2l+1}\,i^{l+1}\left[\frac{e^{-i\left(kr-\frac{l\pi}{2}\right)}}{kr} - \frac{e^{i\left(kr-\frac{l\pi}{2}\right)}}{kr}\right]Y_{l,0}$$

Same for as a incoming and outgoing wave =>

$$\psi_t = \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\,i^{l+1}\left(e^{-i\left(kr-\frac{l\pi}{2}\right)} - \eta_l e^{i\left(kr-\frac{l\pi}{2}\right)}\right)Y_{l,0}$$

2) Number of reactions (incoming - outgoing)

$$-\int \vec{e}_r \vec{j}_t r^2 d\Omega = -\frac{\hbar}{2im}\int \left(\psi_t^* \frac{\partial}{\partial r}\psi_t - \psi_t \frac{\partial}{\partial r}\psi_t^*\right) r^2 d\Omega$$

$$\psi_t = \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^{l+1}(e^{-i\left(kr-\frac{l\pi}{2}\right)} - \eta_l e^{i\left(kr-\frac{l\pi}{2}\right)})Y_{l,0}$$

$$\psi_t^* = \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^{-(l+1)}(e^{i\left(kr-\frac{l\pi}{2}\right)} - \eta_l^* e^{-i\left(kr-\frac{l\pi}{2}\right)})Y_{l,0}^*$$

$$\frac{\partial}{\partial r}\psi_t = -\frac{\psi_t}{r} + \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^{l+1}(-ike^{-i\left(kr-\frac{l\pi}{2}\right)} - ik\eta_l e^{i\left(kr-\frac{l\pi}{2}\right)})Y_{l,0}$$

$$\frac{\partial}{\partial r}\psi_t^* = -\frac{\psi_t^*}{r} + \frac{\sqrt{\pi}}{kr}\sum_{l=0}^{\infty}\sqrt{2l+1}\, i^{-(l+1)}(ike^{i\left(kr-\frac{l\pi}{2}\right)} + ik\eta_l^* e^{-i\left(kr-\frac{l\pi}{2}\right)})Y_{l,0}^*$$

$$\psi_t^*\frac{\partial}{\partial r}\psi_t = -\frac{|\psi_t|^2}{r} + \frac{\pi}{k^2r^2}\sum_{l=0}^{\infty}(2l+1)[-ik+ik|\eta_l|^2 - ik\eta_l e^{2i()} + ik\eta_l^* e^{-2i()}]Y_{l,0}Y_{l,0}^*$$

Because $\int Y_{l,0} Y^*_{l',0} d\Omega = \delta_{ll'}$ (orthogonal functions)

$$\psi_t \frac{\partial}{\partial r} \psi_t^* = -\frac{|\psi_t|^2}{r} + \frac{\pi}{k^2 r^2} \sum_{l=0}^{\infty} (2l+1) ik [1 - |\eta_l|^2 - \eta_l^{2i()} + \eta_l^* e^{-2i()}] Y_{l,0} Y^*_{l,0}$$

$$\Rightarrow \psi_t^* \frac{\partial}{\partial r} \psi_t - \psi_t \frac{\partial}{\partial r} \psi_t^* = -\frac{2i\pi}{kr^2} \sum_{l=0}^{\infty} (2l+1)(1 - |\eta_l|^2) Y_{l,0} Y^*_{l,0}$$

$$-\int \vec{e}_r \vec{j}_t r^2 d\Omega = \frac{\hbar \pi}{mk} \sum_{l=0}^{\infty} (2l+1)(1 - |\eta_l|^2)$$

$$|\vec{j}_{in}| = \frac{\hbar k}{m}$$

$$\Rightarrow \sigma = \frac{-\int \vec{e}_r \vec{j}_t r^2 d\Omega}{|\vec{j}_{in}|}$$

Total cross section for reactions:

$$\sigma = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)(1-|\eta_l|^2)$$

or $$\sigma = \frac{\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)T_l$$

Remark: s-waves will dominate at low E because η_l, the scattering probability, goes down !!!

Remark 2: Compound nucleus which can decay

$i + j \rightarrow \gamma + m \quad i(j,\gamma)m$ EM-transition

$i + j \rightarrow o + m \quad i(j,o)m$ Decay

Cross Section for neutron reactions at low E

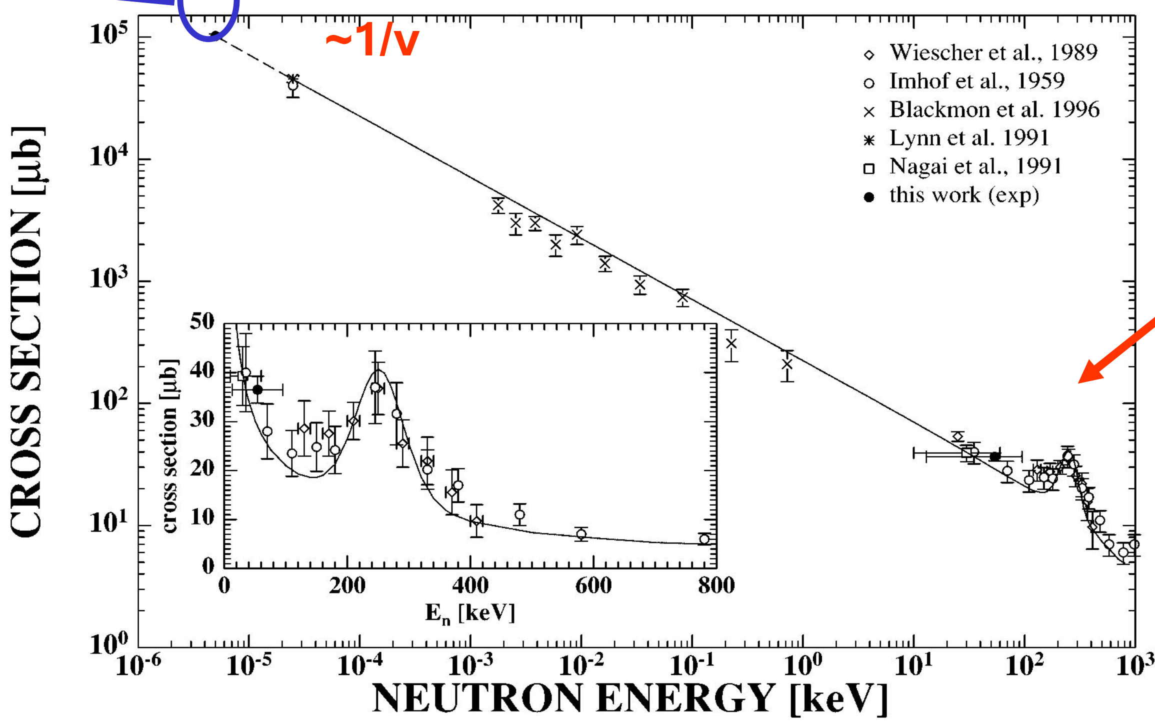
$$k_1 = \frac{\sqrt{2\mu E}}{\hbar} \qquad k_2 = \frac{\sqrt{2\mu(E+Q)}}{\hbar} \approx \text{const}$$

$$\Rightarrow \sigma = \frac{\pi}{k_1^2} \cdot 4\frac{k_1}{k_2} \propto \frac{1}{k_1}$$

$$\sigma \propto \frac{1}{\sqrt{E}} = \frac{1}{v}$$

Example: ^{7}Li(n,γ)

thermal cross section 45.4 mb

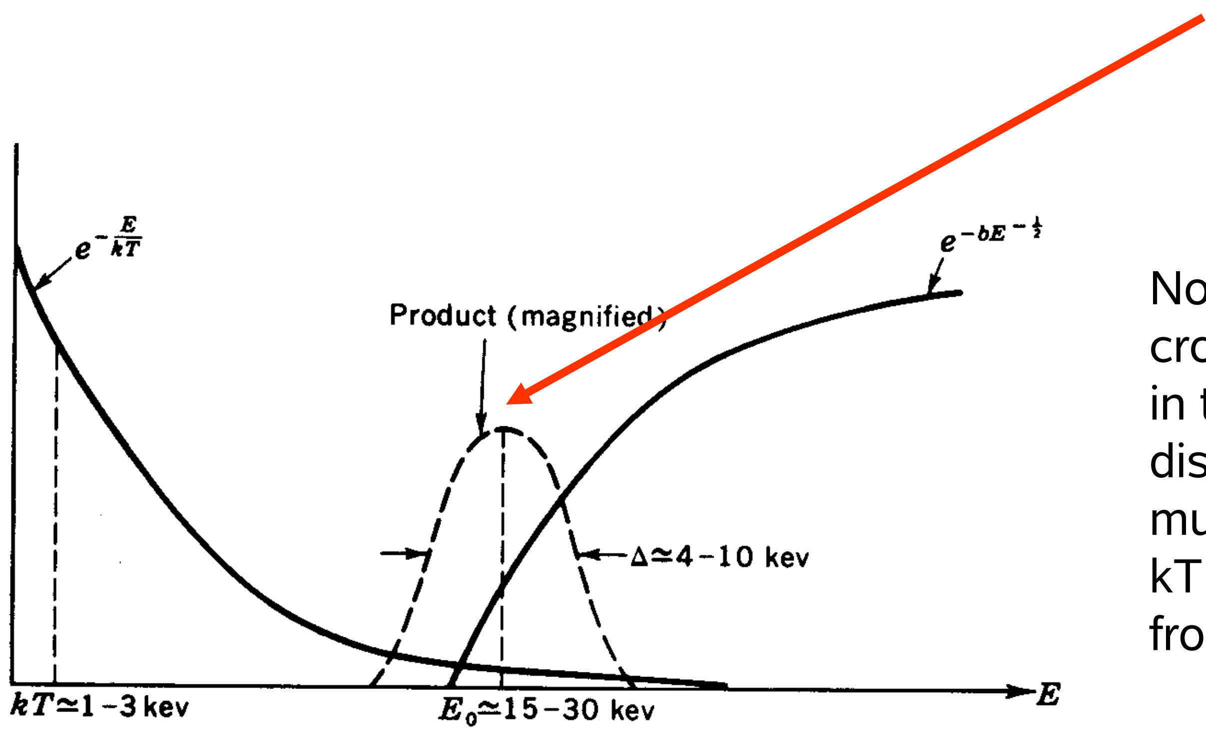


Deviation from 1/v due to resonant contribution

Relevant Reactions for charged particles

$$\sigma = \frac{\pi}{k^2} e^{-2\pi\eta} = \frac{\hbar^2 \pi}{2\mu E} e^{-2\pi\eta}$$

$$\eta = \sqrt{\frac{\mu}{2E}} \frac{Z_i Z_j e^2}{\hbar}$$



Note: relevant cross section in tail of M.B. distribution, much larger than kT (very different from n-capture !)

The S-Factor for Direct Transitions (+EM)

Definition: $S(E) = \sigma E e^{2\pi\eta}$

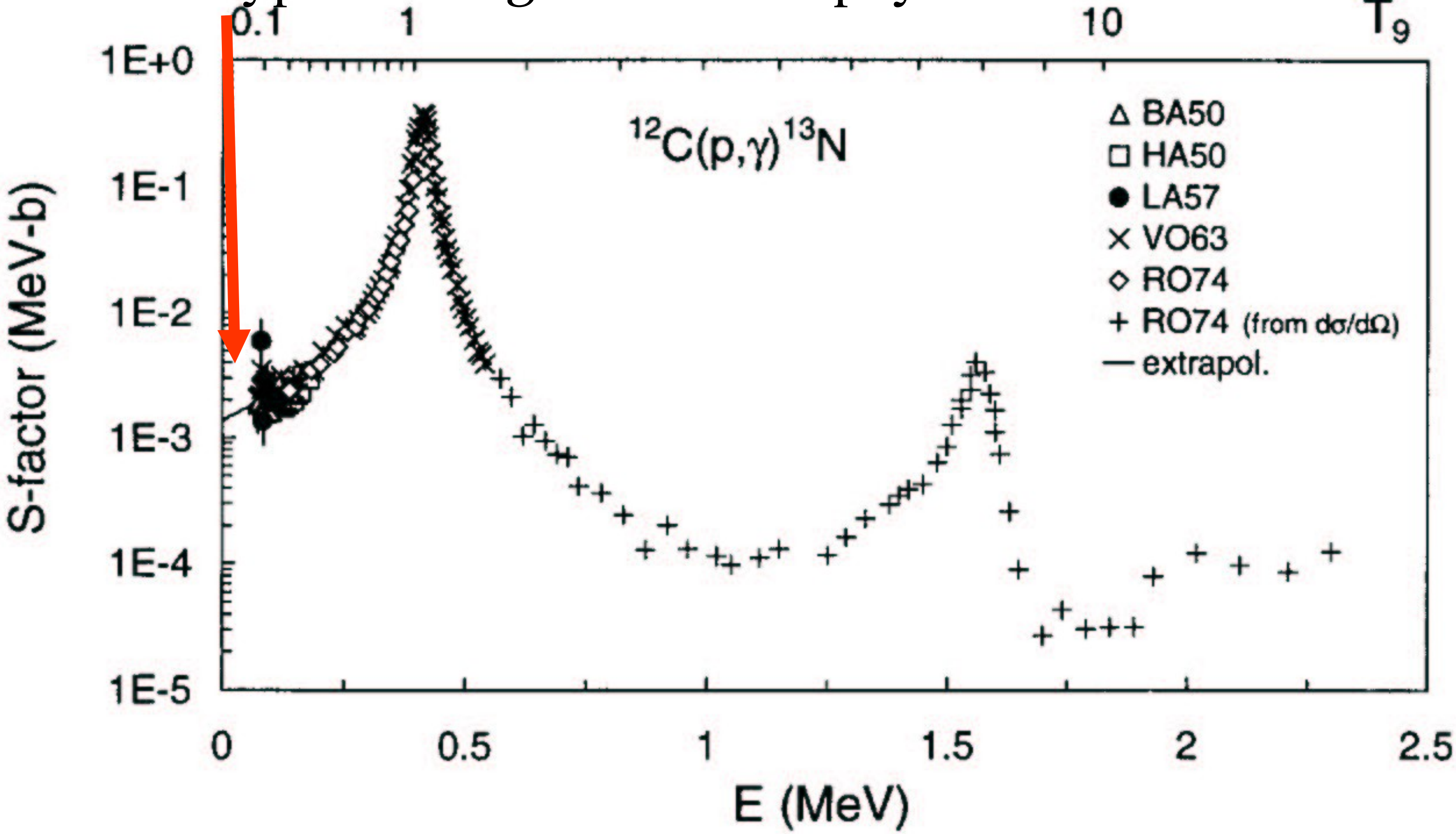
$$\sigma = \frac{S(E)}{E}\ \exp(-2\pi\eta)$$

- is constant for l=0 !!!

- for non-explosive burning in Astrophysical context many rates can be fitted by exponential laws !!

S-Factors for Charged Particle Reactions

Typical energies for Astrophysics



From the **NACRE compilation** of charged particle induced reaction rates on stable nuclei from H to Si (Angulo et al. Nucl. Phys. A 656 (1999) 3

Life could be good in Astrophysics if not for
Resonance Reactions in Compond Nuclei

Problem: $i + j \rightarrow o + m \qquad i(j,o)m$

Transition probability for a specific channel:

$$P_o = T_o / \sum_n T_n$$

Two Different Approaches

a) Statistical Model (Hauser-Feshbach)

$$\sigma_i(j,o) = \frac{\pi}{k_j^2} \frac{(1+\delta_{ij})}{(2I_i+1)(2I_j+1)} \sum_{J,\pi} (2J+1) \frac{T_j(E,J,\pi)T_o(E,J,\pi)}{T_{tot}(E,J,\pi)}$$

b) Individual resonance dominates (Breit Wigner)

(Lorenz-Function with live times of states)

$$\sigma(j,k) = \frac{\pi^2}{k_j^2} \frac{(1+\delta_{ij})}{(2I_i+1)(I_j+1)} \sum_{n} (2J_n+1) \frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2 + (\Gamma_n/2)^2}.$$

General Case with correction Terms

$$\sigma_i(j,o)_{HF} = \frac{\pi}{k_j^2}\sum_J (2J+1)\frac{(1+\delta_{ij})}{(2I_i+1)(2I_j+1)}c(j,o,J,\pi)\frac{T_j(E,J,\pi)T_o(E,J,\pi)}{T_{tot}(E,J,\pi)}$$

$$= \langle \sigma_i(j,o)_{BW} \rangle \quad \text{with}$$

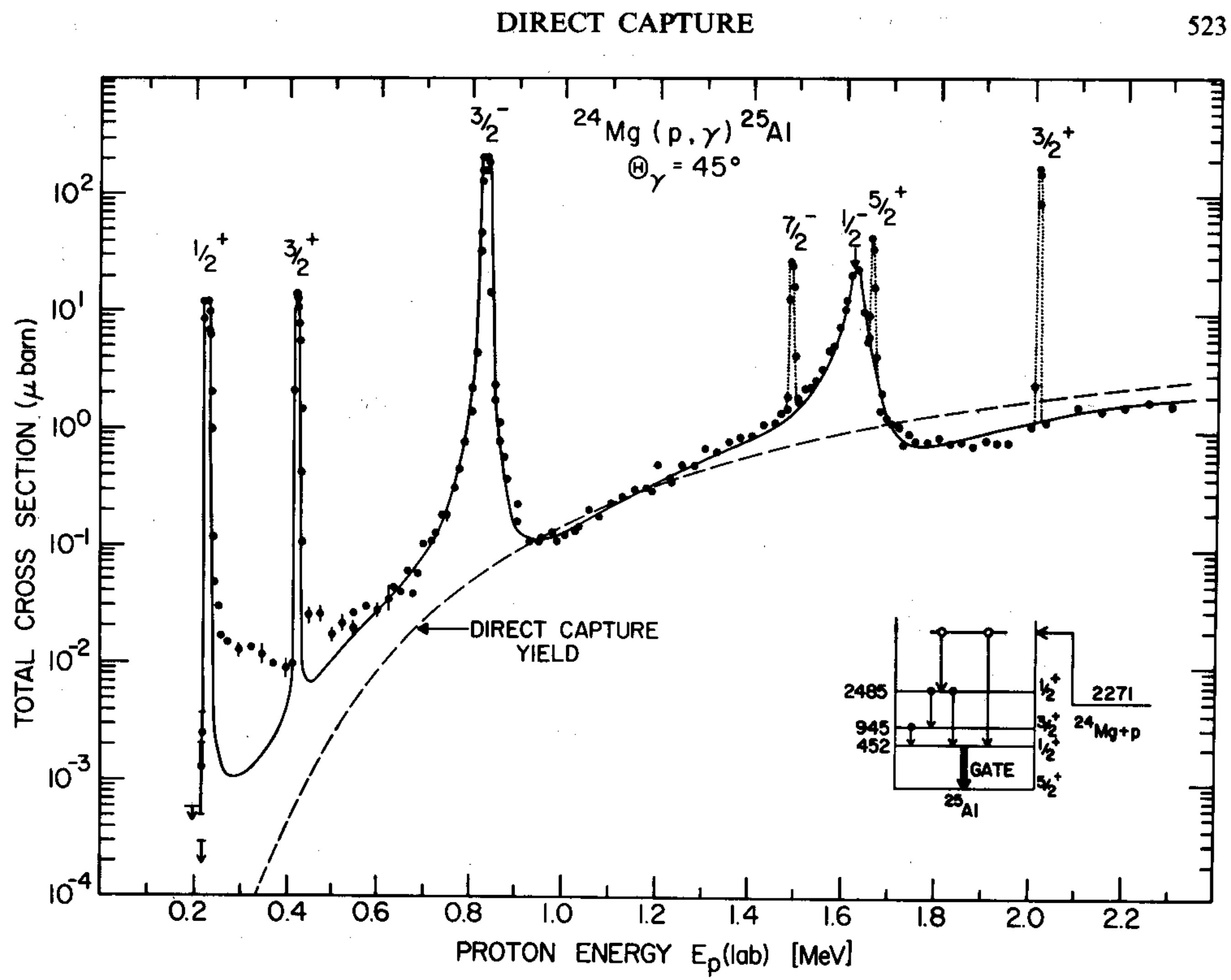
$$\sigma_i(j,o)_{BW} = \frac{\pi}{k_j^2}\sum_n (2J_n+1)\frac{(1+\delta_{ij})}{(2I_i+1)(2I_j+1)}\frac{\Gamma_{j,n}\Gamma_{o,n}}{(E-E_n)^2+(\Gamma_n/2)^2}$$

$$T_j(E,J,\pi) = \frac{2\pi}{D(E,J,\pi)}\left\langle \Gamma_j(E,J,\pi) \right\rangle$$

$$c(j,o,E,J,\pi) = \left\langle \frac{\Gamma_j(E,J,\pi)\Gamma_o(E,J,\pi)}{\Gamma_n(E,J,\pi)} \right\rangle \cdot \frac{\langle \Gamma(E,J,\pi) \rangle}{\langle \Gamma_j(E,J,\pi) \rangle \langle \Gamma_o(E,J,\pi) \rangle}.$$

c is a fluctuation correction (that's the trouble)

Example:



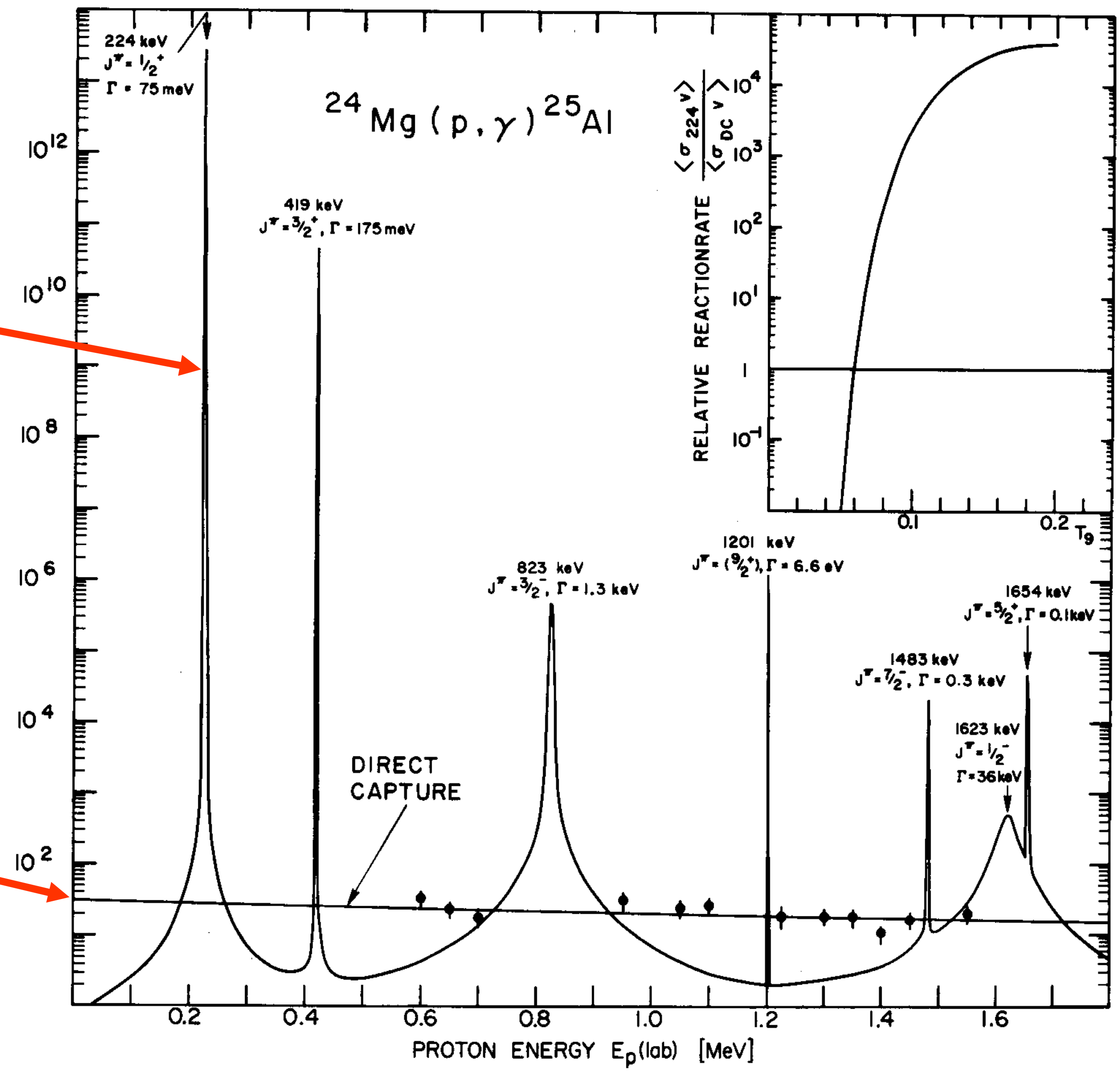
Resonance contributions are on top of direct capture cross sections

… and the corresponding S-factor

Note varying widths !

Not constant S-factor for resonances (log scale !!!!)

~ constant S-factor for direct capture



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