Thermonuclear Reactions

- Reaction Rates

  Theory and Applications

- Nuclear Networks

- Nuclear Statistical Equilibrium

Source: Chapter 3 , and Cameron (1984)

Goal:

Determine the change of abundances and energy

Definition of reaction rate *r*:

$$\sigma = \frac{\text{number of reactions target}^{-1}\text{sec}^{-1}}{\text{flux of incoming projectiles}} = \frac{r/n_i}{n_j v}.$$

Needed:

- cross sections
- velocity distribution of beam and target

Assumption: Non-degenerate Boltzmann gas

Rates

$$r_{i;j} = n_i n_j \int \sigma(|\vec{v}_i - \vec{v}_j|)|\vec{v}_i - \vec{v}_j|\phi(\vec{v}_i)\phi(\vec{v}_j)d^3v_i d^3v_j$$

$$r_{i;j} = n_i n_j \langle \sigma v \rangle_{i;j}$$

with

$$\phi(\vec{v}_i)\phi(\vec{v}_j) = \frac{(m_i m_j)^{3/2}}{(2\pi kT)^3} \exp\left(-\frac{m_i v_i^2 + m_j v_j^2}{2kT}\right)$$

Next step: Transformation into center of mass system (Newton's law)

$$E = \frac{m_i v_i^2}{2} + \frac{m_j v_j^2}{2} = \frac{1}{2} M V^2 + \frac{1}{2} \mu v^2$$

$$\mu = m_i m_j / (m_i + m_j)$$

$$\vec{V} = \frac{m_i \vec{v}_i + m_j \vec{v}_j}{M} \qquad \vec{v} = \vec{v}_i - \vec{v}_j$$

$$\vec{v}_i = \vec{V} + \frac{m_j}{M} \vec{v} \qquad \vec{v}_j = \vec{V} - \frac{m_i}{M} \vec{v}$$

$$\Rightarrow \phi(\vec{v}_i)\phi(\vec{v}_j) = \left(\frac{M}{2\pi kT}\right)^{3/2} \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left(-\frac{MV^2}{2kT} - \frac{\mu v^2}{2kT}\right)$$

$$= \phi(\vec{V})\phi(\vec{v}).$$

$$d^3 v_i d^3 v_j = dv_{i,x} dv_{i,y} dv_{i,z} dv_{j,x} dv_{j,y} dv_{j,z}$$

$$d^3 V d^3 v = dV_x dV_y dV_z dv_x dv_y dv_z$$

$$\langle \sigma v \rangle = \int \sigma(v) v \left(\frac{\mu}{2\pi kT}\right)^{3/2} \exp\left(-\frac{\mu v^2}{2kT}\right) d^3 v$$

$$\langle \sigma v \rangle (T) = \left(\frac{8}{\mu \pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty E \sigma(E) \exp(-E/kT) dE$$

Applications: Determination of Reaction Rates

a) Resonant reactions (often valid for low T)

$$\langle \sigma v \rangle_{res,n} \approx \hbar^2 \left(\frac{2\pi}{\mu k T} \right)^{3/2} \frac{(2J_n + 1)(1 + \delta_{ij})}{(2I_i + 1)(2I_j + 1)} \frac{\Gamma_{j,n} \Gamma_{o,n}}{\Gamma_n} \exp\left(-\frac{E_n}{kT} \right)$$

b) Non-resonant reactions for Neutrons (s-waves with l=0)

$$\sigma_n = \frac{1}{v} S(E) \approx \frac{S(0)}{v} \left(1 + \frac{\dot{S}(0)}{S(0)} E^{1/2} + \frac{1}{2} \frac{\ddot{S}(0)}{S(0)} E \right)$$

$$\langle \sigma v \rangle_{nonres} = S(0) \left(1 + \frac{\dot{S}(0)}{S(0)} \frac{2}{\sqrt{\pi}} (kT)^{1/2} + \frac{\ddot{S}(0)}{S(0)} \frac{3}{4} kT \right)$$

Applications: Determination of Reaction Rates

c) Non-resonant reactions for charged particles

$$\langle \sigma v \rangle_{nonres} = \left(\frac{2}{\mu}\right)^{1/2} S_{eff}(0) \frac{\Delta}{(kT)^{3/2}} \exp(-\tau)$$

$$S_{eff} = S(0)\left[1 + \frac{5}{12\tau} + \frac{S'(0)}{S(0)}\left(E_0 + \frac{35}{36}kT\right) + \frac{1}{2}\frac{S''(0)}{S(0)}\left(E_0^2 + \frac{89}{36}E_0 kT\right)\right]$$

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} = \left(\frac{\pi}{\hbar}\left(\frac{\mu}{2}\right)^{1/2} Z_i Z_j e^2 kT\right)^{2/3}$$

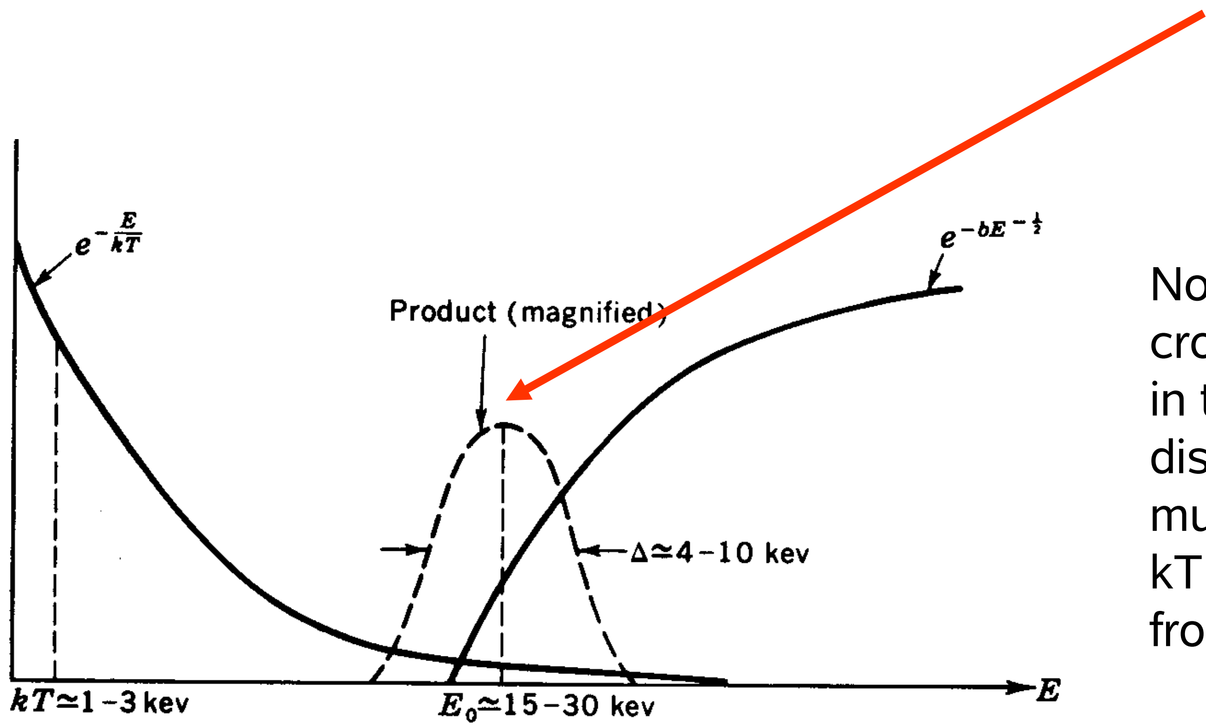
$$\tau = \frac{3E_0}{kT} = 3\left(\frac{b}{2}\right)^{2/3} \frac{1}{(kT)^{1/3}} = 3\left(\frac{\pi}{\hbar}\left(\frac{\mu}{2}\right)^{1/2} Z_i Z_j e^2\right)^{2/3} (kT)^{-1/3}$$

$$\Delta = \frac{4}{\sqrt{3}}(E_0 kT)^{1/2} = \frac{4}{3^{1/2}}\left(\frac{b}{2}\right)^{1/3} kT^{5/6} = \frac{4}{\sqrt{3}}\left(\frac{\pi}{\hbar}\left(\frac{\mu}{2}\right)^{1/2} Z_i Z_j e^2\right)^{1/3} (kT)^{5/6}$$

Relevant Reactions for charged particles

$$\sigma = \frac{\pi}{k^2} e^{-2\pi\eta} = \frac{\hbar^2 \pi}{2\mu E} e^{-2\pi\eta}$$

$$\eta = \sqrt{\frac{\mu}{2E}} \frac{Z_i Z_j e^2}{\hbar}$$



Note: relevant cross section in tail of M.B. distribution, much larger than kT (very different from n-capture !)

Cross Sections of Reverse Reactions $i(j,o)m$

$$\frac{\sigma_i(j,o)_J}{\sigma_m(o,j)_J} = \frac{1+\delta_{ij}}{1+\delta_{om}} \frac{g_o g_m}{g_i g_j} \frac{k_o^2}{k_j^2},$$

Follows directly from symmetry of incoming to outgoing wave and detailed balancing (Blatt and Weisskopf, 1952).

Reactions with non-Boltmann Components
a) Photodisintegration

Photons follow the Boson statistics (no Boltzmann gas)-> LT

$$\lambda_{i;\gamma,o} = \frac{1}{\pi^2 c^2 \hbar^3} \int_0^\infty \frac{g_o g_m}{(1+\delta_{om}) g_i} c^2 \frac{\mu_{om} E_{om}}{E_\gamma^2} \sigma_m(o,\gamma;E_{om}) E_\gamma^2 \exp(-E_\gamma/kT) dE_\gamma$$

$$= \frac{1}{\pi^2 \hbar^3} \frac{g_o g_m}{g_i} \mu_{om} \exp(-Q_{o,\gamma}/kT)$$

$$\times \int_0^\infty E_{om} \sigma_m(o,\gamma;E_{om}) \exp(-E_{om}/kT) dE_{om}$$

$$\lambda_{i;\gamma,o}(T) = \frac{g_o G_m}{(1+\delta_{om}) G_i} \left(\frac{\mu_{om} kT}{2\pi\hbar^2} \right)^{3/2} \exp(-Q_{o,\gamma}/kT) \langle \sigma v \rangle_{m;o,\gamma}$$

Trick for derivation: Relation between forward and backward reaction $m(o,\nu)i$

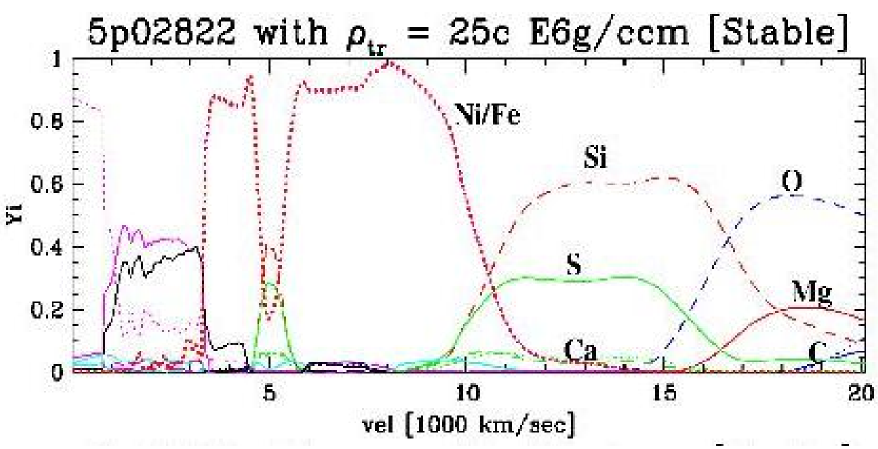
b) Electron Capture at high densities:

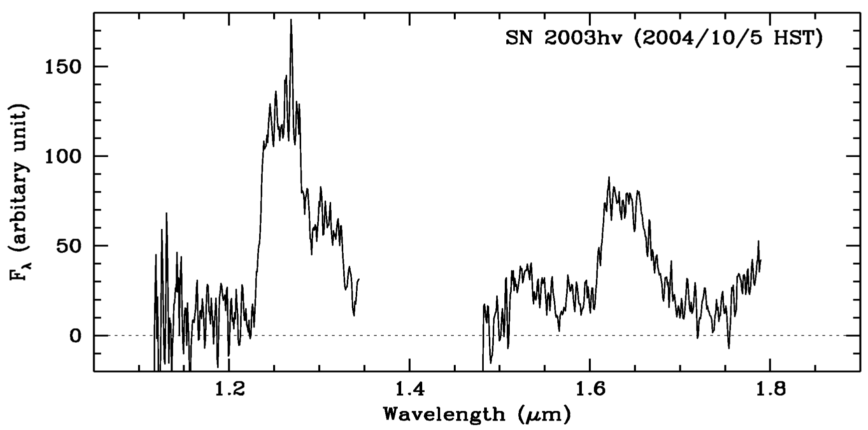
e is degenerate Fermion gas

$$E_F = \frac{\hbar^2}{2m_e}(3\pi^2 N_A)^{2/3}(\rho Y_e)^{2/3}$$

Example: E(1E7g/ccm)=0.75MeV
E(1E9g/ccm)= 4.0 MeV

Electron capture can take place at high densities
Important in Thermonuclear Supernovae, Neutron stars, etc.

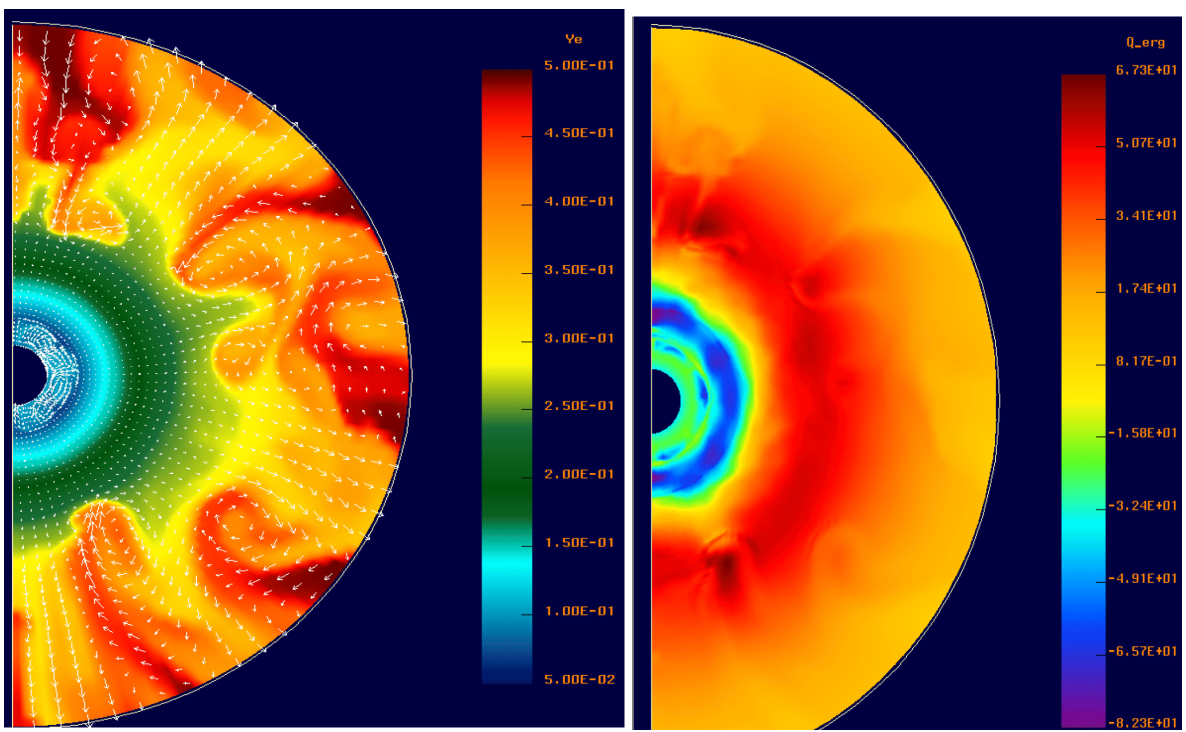




c) Inelastic Neutrino Scattering

$$(Z,A)^* \rightarrow \begin{cases} (Z-1,A-1)+p \\ (Z,A-1)+n \\ (Z-2,A-4)+\alpha. \end{cases}$$

Important in Neutron Stars, Type II Supernovae



e.g. Janka&Mueller (1998)

Decays: $r_i = \lambda_i n_i$

Nuclear Reaction Networks I: i(j,o)m

Thermonuclear Reactions: $$r_{i;j} = \frac{1}{1+\delta_{ij}} n_i n_j \left\langle \sigma v \right\rangle$$

$$\dot{Y}_i = \frac{1}{\rho N_A}\left(\frac{\partial n_i}{\partial t}\right)_\rho = -\frac{r_{i;j}}{\rho N_A} = -\frac{1}{1+\delta_{ij}} \rho N_A \left\langle \sigma v \right\rangle_{i;j} Y_i Y_j$$

$$\dot{Y}_j = \frac{-1}{1+\delta_{ij}} \rho N_A \left\langle \sigma v \right\rangle_{i;j} Y_i Y_j$$

$$\dot{Y}_o = \frac{1}{1+\delta_{ij}} \rho N_A \left\langle \sigma v \right\rangle_{i;j} Y_i Y_j$$

$$\dot{Y}_m = \frac{1}{1+\delta_{ij}} \rho N_A \left\langle \sigma v \right\rangle_{i;j} Y_i Y_j .$$

Nuclear Reaction Networks II: i(j,o)m

Decays, photo, and captures: $r_{i;j} = \frac{1}{1+\delta_{ij}} n_i n_j \langle \sigma v \rangle$

$$\dot{Y}_i = (\frac{\dot{n}_i}{\rho N_A})_\rho = -\frac{r_i}{\rho N_A}$$

$$\Rightarrow \dot{Y}_i = -\lambda_i Y_i \quad \dot{Y}_m = \lambda_i Y_i$$

General Networks

$$(\frac{\partial n_i}{\partial t})_{\rho=const} = \sum_i N_j^i r_j + \sum_{i.k} N_{j,k}^i r_{j;k}$$

Example: Hydrogen Burning as PP I

$$^{1}\mathrm{H}(p,e^{+}\nu_e)^{2}\mathrm{H}$$
$$^{2}\mathrm{H}(p,\gamma)^{3}\mathrm{He}$$
$$^{3}\mathrm{He}(^{3}He,2p)^{4}\mathrm{He}$$

$$\dot{Y}_1 = -\frac{2}{2}\rho N_A \langle 1,1\rangle Y_1^2 - \rho N_A \langle 1,2\rangle Y_1 Y_2 + \frac{2}{2}\rho N_A \langle 3,3\rangle Y_3^2$$
$$\dot{Y}_2 = \frac{1}{2}\rho N_A \langle 1,1\rangle Y_1^2 - \rho N_A \langle 1,2\rangle Y_1 Y_2$$
$$\dot{Y}_3 = \rho N_A \langle 1,2\rangle Y_1 Y_2 - \frac{2}{2}\rho N_A \langle 3,3\rangle Y_3^2$$
$$\dot{Y}_4 = \frac{1}{2}\rho N_A \langle 3,3\rangle Y_3^2.$$

Nuclear Statistical Equilibrium

Assumptions: t(nuc) << t(hyd), Boltzmann Gas

Chemical Equilibrium:
$$\bar{\mu}(Z,N) + \bar{\mu}_n = \bar{\mu}(Z,N+1)$$
$$\bar{\mu}(Z,N) + \bar{\mu}_p = \bar{\mu}(Z+1,N)$$

$$Y(Z,N) = G_{Z,N}(\rho N_A)^{A-1}\frac{A^{3/2}}{2^A}\left(\frac{2\pi\hbar^2}{m_u kT}\right)^{\frac{3}{2}(A-1)} \cdot \exp(B_{Z,N}/kT)Y_n^N Y_p^Z$$

$$\sum_i A_i Y_i = 1$$
$$\sum_i Z_i Y_i = Y_e$$

Result does not depend on rates !!!