Therefore, the kernel for the present situation can be written

$$K(b,a) = \left(\frac{m\omega}{2\pi\hbar i \sin \omega T}\right)^{1/2} \int_a^b \left(\exp \frac{i}{\hbar} \left\{\frac{m}{2} \int_{t_a}^{t_b} \dot{\mathbf{x}}(t)^2\,dt - \frac{1}{m\omega \sin \omega T} \int_{t_a}^{t_b} \int_{t_a}^{t} g[\mathbf{x}(t),t]g[\mathbf{x}(s),s] \sin \omega(t_a - t) \sin \omega(s - t_a)\,ds\,dt\right\}\right) \mathcal{D}\mathbf{x}(t) \qquad (3\text{-}82)$$

with a similar (but more complicated) expression for arbitrary $\mathbf{X}_a$, $\mathbf{X}_b$.

This is a more complicated path integral than any we have had to solve so far. It is not possible to proceed further with the solution until various methods of approximation have been developed in succeeding chapters. Note that the integrand of this path integral can still be thought of as being of the form $e^{(i/\hbar)S}$, but now S is no longer a function of only $\dot{\mathbf{x}}$, $\mathbf{x}$, and t. Instead, S contains a product of variables defined at two different times, s and t. The separation of past and future can no longer be made. This happens because the variable $\mathbf{x}$ at some previous time affects the oscillator which, at some later time, reacts back to affect $\mathbf{x}$. No wave function $\psi(\mathbf{x},t)$ can be defined to give the amplitude that the particle is at some particular place $\mathbf{x}$ at a particular time t. Such an amplitude would be insufficient for continuing calculations into the future, since at any time one must also know what the oscillator is doing.

3-11 EVALUATION OF PATH INTEGRALS BY FOURIER SERIES

Consider the path integral for the harmonic oscillator problem (Prob. 3-8). This is

$$K(b,a) = \int_a^b \left(\exp \left\{\frac{i}{\hbar} \int_{t_a}^{t_b} \left[\frac{m}{2}(\dot{x}^2 - \omega^2 x^2)\,dt\right]\right\}\right) \mathcal{D}x(t) \qquad (3\text{-}83)$$

Using the methods of Sec. 3-5 this path integral can be reduced to a product of two functions, as in Prob. 3-8. The more important of these two functions depends upon the classical orbit for a harmonic oscillator and is given in Eq. (3-59). The remaining function depends upon the time interval only and is written down in Eq. (3-60). This latter function can be written as

$$F(T) = \int_0^0 \left\{\exp\left[\frac{i}{\hbar} \int_0^T \frac{m}{2}(\dot{y}^2 - \omega^2 y^2)\,dt\right]\right\} \mathcal{D}y(t) \qquad (3\text{-}84)$$

We shall solve this, at least to within a factor independent of ω, by a method which illustrates still another way of handling path integrals. Since all paths $y(t)$ go from 0 at $t = 0$ to 0 at $t = T$, such paths can be written as a Fourier sine series with a fundamental period of T. Thus

$$y(t) = \sum_n a_n \sin \frac{n\pi t}{T} \tag{3-85}$$

It is possible then to consider the paths as functions of the coefficients of a_n instead of functions of y at any particular value of t. This is a linear transformation whose jacobian J is a constant, obviously independent of ω, m, and $\hbar$.

Of course, it is possible to evaluate this jacobian directly. However, here we shall avoid the evaluation of J by collecting all factors which are independent of ω (including J) into a single constant factor. We can always recover the correct factor at the end, since we know the value for $\omega = 0$, $F(T) = \sqrt{m/2\pi i\hbar T}$ (a free particle).

The integral for the action can be written in terms of the Fourier series of Eq. (3-85). Thus the kinetic-energy term becomes

$$\begin{aligned}\int_0^T \dot{y}^2\,dt &= \sum_n \sum_m \frac{n\pi}{T}\frac{m\pi}{T} a_n a_m \int_0^T \cos\frac{n\pi t}{T}\cos\frac{m\pi t}{T}\,dt \\ &= T\cdot\frac{1}{2}\sum_n \left(\frac{n\pi}{T}\right)^2 a_n^2 \end{aligned}\tag{3-86}$$

and similarly the potential-energy term is

$$\int_0^T y^2\,dt = T\cdot\frac{1}{2}\sum_n a_n^2 \tag{3-87}$$

On the assumption that the time T is divided into discrete steps of length ϵ as for Eq. (2-19) so that there are only a finite number N of coefficients a_n, the path integral becomes

$$F(T) = J\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\left(\exp\left\{\sum_{n=1}^{N}\frac{im}{2\hbar}\left[\left(\frac{n\pi}{T}\right)^2-\omega^2\right]a_n^2\right\}\right)\frac{da_1}{A}\frac{da_2}{A}\cdots\frac{da_N}{A} \tag{3-88}$$

Since the exponent can be separated into factors, the integral over each coefficient a_n can be done separately. The result of one such integration is

$$\int_{-\infty}^{\infty} \left\{\exp\left[\frac{im}{2\hbar}\left(\frac{n^2\pi^2}{T^2} - \omega^2\right) a_n{}^2\right]\right\} \frac{da_n}{A} = \left(\frac{n^2\pi^2}{T^2} - \omega^2\right)^{-1/2} \tag{3-89}$$

Thus the path integral is proportional to

$$\prod_{n=1}^{N} \left(\frac{n^2\pi^2}{T^2} - \omega^2\right)^{-1/2} = \prod_{n=1}^{N} \left(\frac{n^2\pi^2}{T^2}\right)^{-1/2} \prod_{n=1}^{N} \left(1 - \frac{\omega^2 T^2}{n^2\pi^2}\right)^{-1/2} \tag{3-90}$$

The first product does not depend on ω and combines with the jacobian and other factors we have collected into a single constant. The second factor has the limit $[(\sin \omega T)/\omega T]^{-1/2}$ as $N \to \infty$, that is, as $\epsilon \to 0$. Thus

$$F(T) = C\left(\frac{\sin \omega T}{\omega T}\right)^{-1/2} \tag{3-91}$$

where C is independent of ω. But for $\omega = 0$ our integral is that for a free particle, for which we have already found that

$$F(T) = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \tag{3-92}$$

Hence for the harmonic oscillator we have

$$F(T) = \left(\frac{m\omega}{2\pi i\hbar \sin \omega T}\right)^{1/2} \tag{3-93}$$

which is to be substituted in Eq. (3-59) to obtain the complete solution.

Problem 3-13 By keeping track of all the constants, show that the implication is that the jacobian satisfies

$$J\sqrt{N}\left(\frac{T}{\pi}\right)^N \prod_{n=1}^{N} \frac{1}{n} \to 1 \qquad \text{as } N \to \infty \tag{3-94}$$